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An Isomorphism Between the Para-Fermi Algebra and the Algebra 0(*n*,*n*+1)

T. D. PALEV

Institute of Physics, Bulgarian Academy of Sciences, Sofia, Bulgaria

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Abstract

It is shown that the relations which define the para-Fermi creation and annihilation operators b_i, a_i $(i = 1,\ldots,n)$ can be considered as commutation relations of the algebra $0(n, n+1)$. It turns out that every representation of $0(n, n+1)$ determines a representation of the para-Fermi algebra and vice versa.

In the present note we prove that the relations which define the para-Fermi creation and annihilation operators b_i, a_i $(i = 1, \ldots, n)$ are nothing but the commutation relations of the algebra $0(n, n+1)$, considered in a realisation where also an associative multiplication between the elements of $0(n, n+1)$ is defined. We show that the para-Fermi operators are part of the generators of $0(n, n+1)$ which define uniquely the algebra. As a consequence it turns out that every representation of the para-Fermi algebra defines a representation of the algebra $0(n, n+1)$ and vice versa.

Consider the set of $2n^2 + n$ entities $G = (a_i, b_i, c_{ij}, d_{ij}, e_{ij} / d_{ij} = -d_{ji}, e_{ij} = -e_{ji}, i,j = 1,\ldots,n)$ and let ϕ be an arbitrary field. Denote by $\tilde{U}_n$ the $2n^2 + n$ dimensional vector space spanned on the elements of G over ϕ. For arbitrary a, b, $c \in \tilde{U}_n$ and α, $\beta \in \phi$ define a bilinear antisymmetrical law of composition $[a, b] \in U_n$, i.e.

$$\begin{aligned} [\alpha a + \beta b, c] &= \alpha[a, c] + \beta[b, c] \\ [a, b] &= -[b, a] \end{aligned} \tag{1}$$

by the following relations:*

$$\begin{aligned} &[a_i, b_j] = c_{ij}, \qquad [a_i, a_j] = d_{ij}, \qquad [b_i, b_j] = e_{ij} \\ &[c_{ij}, a_k] = 2\delta_{ij} a_i \\ &[c_{ij}, b_k] = -2\delta_{ik} b_j \end{aligned} \tag{2a}$$

$$\begin{aligned} [e_{ij}, a_k] &= 2\delta_{jk} b_i - 2\delta_{ik} b_j \\ [d_{ij}, b_k] &= 2\delta_{jk} a_i - 2\delta_{ik} a_j \\ [d_{ij}, a_k] &= [e_{ij}, b_k] = 0 \end{aligned} \tag{2b}$$

*Unless otherwise stated, the lower-case indices run from 1 to n.

$$
\begin{aligned}
&[c_{ij}, c_{kl}] = 2\delta_{jk} c_{il} - 2\delta_{il} c_{kj} \\
&[c_{ij}, d_{kl}] = 2\delta_{jl} d_{ki} - 2\delta_{jk} d_{li} \\
&[c_{ij}, e_{kl}] = 2\delta_{il} e_{jk} - 2\delta_{ik} e_{jl} \\
&[d_{ij}, e_{kl}] = 2\delta_{jk} c_{il} + 2\delta_{il} c_{jk} - 2\delta_{ik} c_{jl} - 2\delta_{jl} c_{ik} \\
&[d_{ij}, d_{kl}] = [e_{ij}, e_{kl}] = 0
\end{aligned} \tag{2c}
$$

One can easily verify that with respect to the composition (2) the Jackoby identity holds, so that $\tilde{U}_n$ is a Lie algebra. It is convenient to embed $\tilde{U}_n$ into an associative algebra V_n, so that the composition law of $\tilde{U}_n$ in V_n will have the usual form of a commutator, that is,

$$[a, b] = ab - ba \qquad a, b \in U_n \tag{3}$$

Denote by V_n the associative algebra of all polynomials of the elements from G in which the relations (2) with commutator defined as (3) hold. The algebra V_n is a factor algebra of the free-associative algebra of the set G with respect to the ideal generated by the relations (2) [for more complete discussions see Doebner & Palev (1970)]. The algebra V_n contains a Lie isomorphic image U_n of $\tilde{U}_n$. To obtain this it is enough to consider the abstract composition law (2) as a commutator defined by (3).

The commutation relations (2c) in U_n are consequences of the relations (2a) and (2b). Inserting (2a) in (2b) we obtain the following relations:

$$
\begin{aligned}
&[[a_p, b_q], a_r] = 2\delta_{qr} a_p \\
&[[a_p, b_q], b_r] = -2\delta_{pr} b_q \\
&[[b_p, b_q], a_r] = 2(\delta_{qr} b_p - \delta_{pr} b_q) \\
&[[a_p, a_q], b_r] = 2(\delta_{qr} a_p - \delta_{pr} a_q) \\
&[[a_p, a_q], a_r] = [[b_p, b_q], b_r] = 0
\end{aligned} \tag{4}
$$

We observe that the equations (4) are not only simple consequences of the relations (2), but also they are enough to restore the commutation relations (2), that is, they define uniquely the Lie-algebraic structure in U_n. In these notations the algebra U_n is spanned on the following $2n^2 + n$ linearly independent elements:

$$a_i, b_i, [a_i, b_j], [a_p, a_q], [b_p, b_q] \qquad p < q \tag{5}$$

The entities a_i, b_i which satisfy the equations (4) were first introduced by H. S. Green (1953). They are called para-Fermi annihilation and creation operators. Therefore we call the Lie algebra U_n para-Fermi algebra. This algebra is the minimal Lie sub-algebra of the infinite dimensional associative algebra V_n considered as a Lie algebra which contains the para-Fermi operators. We shall show that the real para-Fermi algebra U_n is isomorphic to the algebra $0(n, n+1)$. In this way we shall prove that the relations (4), which serve as a definition of the para-Fermi operators, are nothing but the commutation relations of $0(n, n+1)$.

We first prove that U_n is a simple Lie algebra. From this result it follows that the complex para-Fermi algebra is isomorphic to one of the classical algebras B_n or C_n, since they are the only complex simple Lie algebras of dimension $2n^2+n$ (up to isomorphisms for $n=1, 2, 3$) [see for instance Pontrjagin (1958)].

Lemma. The para-Fermi algebra U_n is a simple Lie algebra.

Proof. Denote by $U_n{}^1$ and $U_n{}^2$ the subspaces of U_n spanned on the operators a_i, b_j and $[a_i, b_j]$, $[a_i, a_j]$, $[b_i, b_j]$ correspondingly. Then the equations (4) give

$$[U_n{}^1, U_n{}^2] = U_n{}^1 \tag{6}$$

Let r be an arbitrary nonzero element from U_n and let J_r be the Lie ideal generated by r. Take the element a_1. From the relations [see (4)] $[[a_1,b_1],b_1]=-2b_1$, $[[a_j,b_1],a_1]=2a_j$ and $[[b_j,b_1],a_1]=2b_j$, it follows that J_{a_1} contains all para-Fermi operators and hence all generators (5) of U_n. Thus $J_{a_1}=U_n$. In a similar way one can prove that $J_{a_i}=J_{b_i}=U_n$. Therefore for any $0\neq r\in U_n$ we have $U_n=J_{r_1}$. Let $0\neq r_2\in U_n{}^2$. Then $[r_2, U_n{}^1]\cap U_n{}^1\neq 0$ and hence $J_{r_2}=U_n$. For an arbitrary $r=r_1+r_2$, $r_i\in U_n{}^i$, $r_i\neq 0$, $i=1, 2$, $U_n{}^1\in [r_1+r_2, r_2]\neq 0$. So we obtain:

$$J_r = U_n \qquad r\neq 0 \tag{7}$$

We have shown that the intersection of all ideals which contain an arbitrary nonzero element $r\in U_n$ coincides with U_n. Hence the para-Fermi algebra U_n contains no nontrivial ideals, i.e. it is simple.

The only complex simple Lie algebras of dimension $2n^2+n$ are the classical algebras B_n and C_n. Therefore the real para-Fermi algebra is isomorphic to one of the real forms of these algebras. It turns out that ($\sim$ denotes isomorphism)

$$U_n \sim 0(n, n+1) \tag{8}$$

and to prove this we construct the isomorphism in an explicit form.

First we introduce some convenient notations and choose a proper basis in $0(n,n+1)$. Let β be a square $2n+1$-dimensional diagonal matrix with $n(n+1)$ elements equal to 1 (−1). We put

$$\beta = \begin{pmatrix} 1 & & & & & & & \\ & -1 & & & & & & \\ & & 1 & & & & & \\ & & & -1 & & & & \\ & & & & \ddots & & & \\ & & & & & 1 & & \\ & & & & & & -1 & \\ & & & & & & & -1 \end{pmatrix} \tag{9}$$

The algebra $0(n,n+1)$ can be defined as the set of all square $2n+1$-dimensional matrices γ which satisfy the matrix equation [see for instance Helgason (1962)]

$$\overset{\tau}{\gamma}\beta + \beta\gamma = 0 \qquad \tau\text{-transposition} \tag{10}$$

Denote by e_s^r $(r, s = 1, \ldots, 2n+1)$ a square $2n+1$-dimensional matrix with all elements equal to zero except the element on the cross of the rth row and sth column which is equal to 1. Then it is possible to choose the following basis in the matrix representation defined by equation (10) of $0(n,n+1)$:

$$\begin{aligned} f_1{}^p &= e_{2n+1}^{2p-1} + e_{2p-1}^{2n+1} & f_2{}^p &= e_{2n+1}^{2p} - e_{2p}^{2n+1} \\ f_3{}^p &= e_{2p}^{2p-1} + e_{2p-1}^{2p} & f_4{}^{pq} &= e_{2p}^{2q} - e_{2q}^{2p} \\ f_5{}^{pq} &= e_{2q}^{2p-1} + e_{2p-1}^{2q} & f_6{}^{pq} &= e_{2q-1}^{2p-1} - e_{2p-1}^{2q-1} \\ f_7{}^{pq} &= e_{2q-1}^{2p} + e_{2p}^{2q-1} \end{aligned} \tag{11}$$

where $p, q = 1, \ldots, n$ and $p < q$. Introduce also a new basis in the para-Fermi algebra U_n $(p < q; p, q = 1, \ldots, n)$:

$$\begin{aligned} g_1{}^p &= \tfrac{1}{2}(a_p + b_p) \\ g_2{}^p &= \tfrac{1}{2}(a_p - b_p) \\ g_3{}^p &= \tfrac{1}{2}[a_p, b_p] \\ g_4{}^{pq} &= \tfrac{1}{4}([a_p, a_q] + [b_p, b_q] - [a_p, b_q] + [a_q, b_p]) \\ g_5{}^{pq} &= \tfrac{1}{4}(-[a_p, a_q] + [b_p, b_q] + [a_p, b_q] + [a_q, b_p]) \\ g_6{}^{pq} &= \tfrac{1}{4}([a_p, a_q] + [b_p, b_q] + [a_p, b_q] - [a_q, b_p]) \\ g_7{}^{pq} &= \tfrac{1}{4}([a_p, a_q] - [b_p, b_q] + [a_p, b_q] + [a_q, b_p]) \end{aligned} \tag{12}$$

In these notations the one-to-one linear mapping θ of the real para-Fermi algebra onto $0(n,n+1)$, which preserves the commutation relations, can be defined in the following way $(i = 1, 2, 3; j = 4, 5, 6, 7)$

$$\theta g_i{}^p = f_i{}^p, \qquad \theta g_j^{pq} = f_j^{pq} \tag{13}$$

One can verify this by simple, but rather long, calculations using the equations (4) and the commutation relations $[e_j{}^i, e_l{}^k] = \delta_{jk} e_l{}^i - \delta_{il} e_j{}^k$. Thus we have proved the following theorem:

Theorem. The real para-Fermi algebra U_n is isomorphic to the algebra $0(n,n+1)$. Over the field of the complex numbers U_n is isomorphic to the algebra B_n.

The compact real form $0(2n+1)$ of B_n is spanned on the elements $i^n f_n$ $(n = 1, \ldots, 7)$.

From the theorem we can draw some important conclusions concerning the representations of the para-Fermi operators. The mapping $\tau: a_i \to A_i$, $b_i \to B_i$ of a_i, b_i onto a set A_i, B_i of linear operators in a Hilbert space defines a representation of the para-Fermi operators if A_i, B_i satisfy the equations (4) with commutator (3). In this case the operators

$$A_i, B_i, [A_i, B_j], [A_p, A_q], [B_p, B_q] \qquad p < q \tag{14}$$

span a basis of a representation of $0(n,n+1)$. On the contrary. Let the mapping τ define a linear representation of the algebra $0(n,n+1)$. Denote by A_i, B_i, C_{ij}, D_{ij} and E_{ij} the images of the generators of the algebra under τ which are chosen to satisfy the commutation relations (2). Then the operators $A_i = \tau a_i$, $B_i = \tau b_i$ give a representation of the para-Fermi algebra. Moreover, because of (14) the representation of the para-Fermi algebra and of $0(n,n+1)$ which correspond to each other are simultaneously reducible or irreducible. Thus we have deduced the following corrolary of the theorem.

Corollary. The operators A_i, B_j ($i, j = 1, \ldots, n$) define an (irreducible) representation of the para-Fermi operators a_i, b_j if and only if the operators (14) span a basis of an (irreducible) representation of the algebra $0(n,n+1)$.

In this way the problem to find all representations of the para-Fermi operators reduces mainly to the determination of the representations of the classical algebra $0(n,n+1)$.

In conclusion we want to point out that the isomorphism between U_n and $0(n,n+1)$ does not depend on the order of the para-statistics of the para-Fermi operators. The representations of any order of para-statistics (in particular of the Fermi creation and annihilation operators) are contained among the representations of $0(n,n+1)$. The set S of all representations of the para-Fermi operators with arbitrary order of the para-statistics does not exhaust all representations since S contains only a countable set of finite-dimensional representations, whereas the class of the representations of the algebra $0(n,n+1)$ is non-countable and contains also infinite-dimensional representations.

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